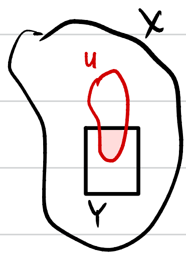
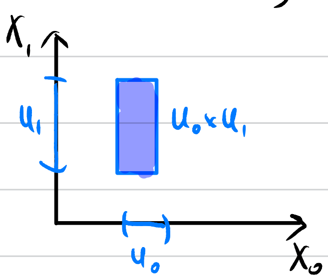
Descriptive Set Theory
Lecture 2

Relative topology. Let X be a top. space and $Y \subseteq X$. Then the *relative topology* on Y is defined by declaring sets of the form $U \cap Y$ open, for $U \subseteq X$ open.



Product topology / top. of ptwise convergence.

For top. spaces X_0 & X_1, the product top. on $X_1 \times X_2$ is defined by declaring open the *open rectangles*, i.e. sets of the form $U_0 \times U_1$, where $U_i \subseteq X_i$ is open. That is, the topology is generated by open rectangles, in fact, the open rectangles form a basis, so every open set in $X_0 \times X_1$ is a union of open rectangles.



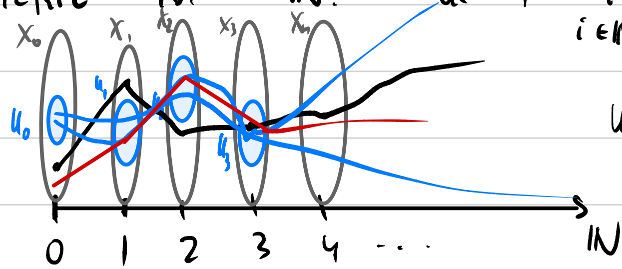
Similarly, for $X_0 \times X_1 \times \dots \times X_n$, the open rectangles $U_0 \times U_1 \times \dots \times U_n$ form a basis. What about infinite products?

Let I be a typically infinite index set, e.g. $I := \mathbb{N}, \mathbb{R}$.

For each $i \in I$, let X_i be a top. space. Then $\prod_{i \in I} X_i$ is the set of sequences $(x_i)_{i \in I}$ s.t. $x_i \in X_i$ $\forall i \in I$.

The open rectangles in this case are sets of the form $U_0 \times U_1 \times \dots \times U_n \times X_{n+1} \times X_{n+2} \times \dots$ (assuming $I = \mathbb{N}$).

I think of elements of $\prod_{i \in I} X_i$ as functions $f : I \to \bigcup_{i \in I} X_i$ s.t. $f(i) \in X_i$ $\forall i \in I$. E.g., if $I = \mathbb{R}$ and $X_i = \mathbb{R}$ $\forall i \in I$, then $\prod_{i \in \mathbb{R}} \mathbb{R} = \mathbb{R}^{\mathbb{R}}$ = all function $\mathbb{R} \to \mathbb{R}$. Let's draw the picture for $I = \mathbb{N}$: let $f \in \prod_{i \in \mathbb{N}} X_i$. The "snakes" that go through all hoops U_0, U_1, U_2, U_3 for an open rectangle (aka cylindrical set).



For $f \in \prod_{i \in \mathbb{N}} X_i$, we write $f(i)$ for the ith coordinated.

Observation. Product top is the topology of pointwise convergence, i.e. for a sequence $(f_n) \subseteq \prod_{i \in I} X_i$ and $f \in \prod_{i \in I} X_i$,

$$f_n \xrightarrow{\text{prod. top.}} f \iff f_n(i) \to f(i) \text{ as } n \to \infty, \ \forall i \in I.$$

Proof. Homework.

Tychonoff's theorem. Products of compact spaces are compact.

DST begins

Def. A top. space X is called **Polish** if it's 2^{nd}-ctbl and completely metrizable (<=> separable and completely metrizable).

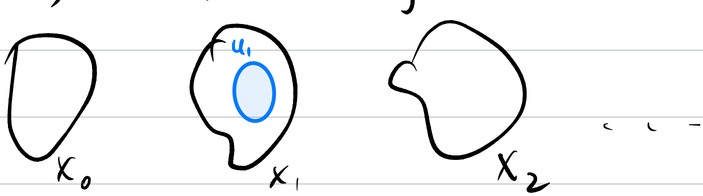
Examples.
- $\mathbb{N}$ or finite sets with discrete top.
- $\mathbb{R}^d$, $\mathbb{C}^d$, $d \in \mathbb{N}$.
- ℓ^p and $L^p(\mathbb{R}, \lambda)$.

Obs. All closed subsets of a Polish are Polish in the relative top.

Polish operations. (a) Ctbl disjoint unions of Polish spaces are Polish.

(b) ctbl products of Polish spaces are Polish.

Proof. (a) Let $X_0, X_1, \dots$ be Polish spaces and suppose that they are pairwise disjoint.



The top. space $X = \bigsqcup_{n \in \mathbb{N}} X_n$ is formed by declaring open any $U_n \subseteq X_n$ open, in particular $U_n := X_n$.

X is still 2^{nd} ctbl because ctbl unions of ctbl sets are still ctbl.

Observation. For every metric d on a set X, the following metric generates the same top. and is bdd by 1: $d_1(x_0, x_1) := \min(1, d(x_0, x_1))$.

Let d_n be a complete compatible metric on X_n s.t. $d_n \leq 1$. Then let the distance between any $x_n \in X_n$ and $x_m \in X_m$, $n \neq m$, to be 1. This extends the metric to all of X and it's easy to check that any Cauchy sequence in X is eventually in one X_n (for some $n \in \mathbb{N}$) and hence converges, by the completeness of d_n.

(b) Let d_n be a compl. compat. metric for X_n, $d_n \leq 1$. We define a metric d on $X := \prod_{n \in \mathbb{N}} X_n$ by: for $x, y \in X$,

$$d(x,y) := \sum_{n \in \mathbb{N}} d_n(x(n), y(n)) \cdot 2^{-n}.$$

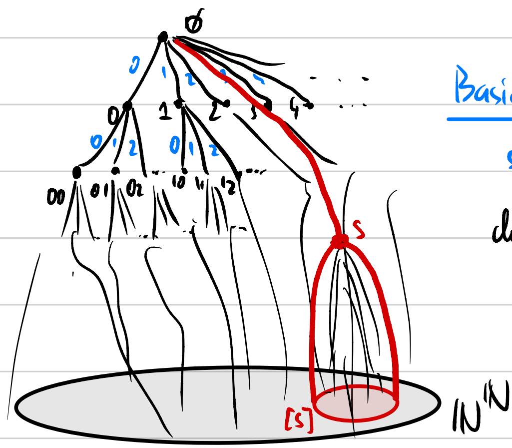
We need to show that this metric is complete and generates the product top. (homework). X is 2^{nd} ctbl because there are only ctbly many open rectangles whose sides are taken from a fix ctbl bases of the respective coordinates. $\square$

Product Polish spaces. (a) $\mathbb{R}^{\mathbb{N}}$

(b) $[0,1]^{\mathbb{N}}$ called the **Hilbert cube**.

(c) $\mathbb{N}^{\mathbb{N}}$, $\mathbb{N}$ discrete, is called the **Baire space**.

Trees. We think of elements of $\mathbb{N}^{\mathbb{N}}$ as infinite branches trough the following tree: $\mathbb{N}^{<\mathbb{N}} := \bigcup_{n\in\mathbb{N}} \mathbb{N}^n$.



Basic open sets.

7	0	1	2	*	*	*	*

$s := (3,0,1,2) \in \mathbb{N}^4$ and the corresponding open rectangle is

$$[s] := \{x \in \mathbb{N}^{\mathbb{N}} : \underbrace{s \subseteq x}_{\forall i<|s|,\ s(i)=x(i)}\}$$

$$[s]^c = \bigcup_{\substack{t\in\mathbb{N}^{|s|} \\ t\neq s}} [t].$$

Thus, $[s]$ is both open and closed, we call such sets **clopen**. We refer to sets of the form $[s]$ as the basic clopen sets/rectangles/cylinders.

A nice compatible (ultra) metric is, for $x, y \in \mathbb{N}^{\mathbb{N}}$,

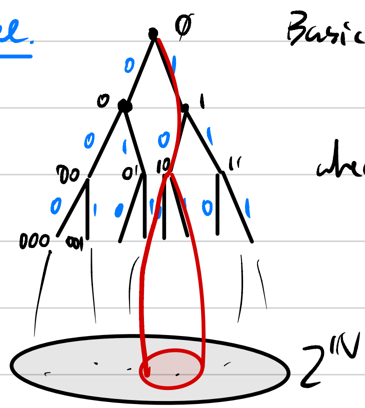
$$d(x,y) := 2^{-n(x,y)}, \text{ where } n(x,y) := \min i \text{ s.t. } x(i) \neq y(i).$$

It is easier to show that this metric generates the prod. top. and is complete, follows from the following:

Obs. For a sequence $(x_n) \subseteq \mathbb{N}^{\mathbb{N}}$, if it is Cauchy then for each index $i \in \mathbb{N}$, $(x_n(i))_{n \in \mathbb{N}}$ is eventually const.

(d) $2^{\mathbb{N}} \subseteq \mathbb{N}^{\mathbb{N}}$ is called the Cantor space. It is a compact subset of the Baire space with relative top.

Corresponding tree.



Basic clopen sets are given by

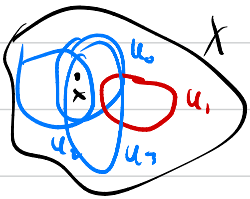
$$[s] := \{x \in 2^{\mathbb{N}} : s \subseteq x\},$$

where $s \in 2^{<\mathbb{N}}$.

Recall that reals have a binary rep., i.e. $\mathbb{R} \hookrightarrow 2^{\mathbb{N}}$. This is true of all 2nd ctbl Hausdorff space, in particular all Polish spaces:

Coding lemma. Any Polish space X injects into $2^{\mathbb{N}}$ and this injection ι is such that $\iota^{-1}(\boxed{*|*|\cdots|*|1|*|\cdots})$ is open.

Proof.



Let (U_n) be a ctbl basis.

Define $\iota : X \to 2^{\mathbb{N}}$ by $x \mapsto (y_n)_{n\in\mathbb{N}}$,

where $y_n := \begin{cases} 0 & \text{if } x \notin U_n \\ 1 & \text{if } x \in U_n \end{cases}$.

Hausdorffness implies that this map is injective.

Moreover, $\iota^{-1}\Big(\underset{0\ \ 1\ \ 2\ \ 3}{\boxed{*|*|*|1|*|*}}\Big) = U_3 \;\ldots$

$\square$